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Note

The Number of Small Semispaces of a Finite Set of Points in the Plane

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For a configuration S of n points in the plane, let $g_k(S)$ denote the number of subsets of cardinality $\leqslant k$ cut off by a line. Let $g_{k,n} = \max\{g_k(S): |S| = n\}$. Goodman and Pollack (*J. Combin. Theory Ser. A* **36** (1984), 101–104) showed that if $k < n/2$ then $g_{k,n} \leqslant 2nk - 2k^2 - k$. Here we show that $g_{k,n} = k \cdot n$ for $k < n/2$. © 1986 Academic Press, Inc.

Let S be a finite set of points in the plane. Following Goodman and Pollack [GP2] we call the intersection of S with a half plane a *semispace* of S. A semispace of S of cardinality k is called a *k-set* of S. Let $f_k(S)$ denote the number of k-sets of S and put $g_k(S) = \sum_{i=1}^{k} f_i(S)$.

Define

$$g_{k,n} = \max\{g_k(S): |S| = n\}.$$

Thus $g_{k,n}$ is the maximal number of $(\leqslant k)$-sets of n points in the plane. Since $g_{k,n} = g_{n-k,n}$ we may restrict our attention to the case $k \leqslant n/2$.

Goodman and Pollack [GP2] considered the problem of estimating $g_{k,n}$ and proved that if $k < n/2$ then $g_{k,n} \leqslant 2nk - 2k^2 - k$.

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In this note we deterime $g_{k,n}$ precisely for all $k < n/2$, by proving:

THEOREM 1. *For* $k < n/2$, $g_{k,n} = k \cdot n$

We have two proofs of Theorem 1; a combinatorial one and a geometric one. Since the first proof is more general we present it in detail and only sketch the second. Our combinatorial proof is based on the ideas of [GP2].

The n vertices of any convex polygon in the plane show that $g_{k,n}$ is at least the quantity mentioned in the theorem. To prove the upper bound we first note that we may assume that the points of S form a simple configuration, i.e., no three points of S are collinear and no two connecting lines (i.e., lines determined by two points of S) are parallel. This follows from the fact that a small perturbation of S will not decrease $g_k(S)$.

Following [GP2] we consider a more general combinatorial problem. We associate with S a sequence of permutations on the n points of S as follows. Choose a directed line L, which is not orthogonal to any connecting line of S, and project the points of S orthogonally onto L. Let P_0 denote the order of these projections on L. Now rotate L counterclockwise. Whenever L passes a direction orthogonal to a connecting line determined by the points $a, b \in S$ the order of the projected points on L is changed by the adjacent transposition (a, b).

After 180° the points fall on L in the reverse order. In this way (after 360°) we obtain a cyclic sequence of permutations $P(S) = P_0, P_1, ..., P_{2N} = P_0$, where $N = \binom{n}{2}$ and

(1) P_i and P_{i+N} are in reverse order (from here on addition of indices is taken modulo $2N$);

(2) P_{k+1} differs from P_k by an adjacent transposition (=switch).

Note that a k-set of S occurs as an initial k-segment of some P_i (and hence as a terminal k-segment of P_{i+N}). As a matter of fact $f_k(S)$ is precisely the number of switches in position k in P, i.e., the number of switches between the kth and the $(k+1)$st indices, since each such switch creates exactly one new k-set. This number equals, of course, the number of switches in position $n-k$ in P.

Call a sequence of permutations P satisfying (1) and (2) an *n-sequence*. (Note that in [GP2] an n-sequence is half of our n-sequence.) For $k \leq n/2$ let $F_k(P)$ denote the number of switches in position k in P, put $G_k(P) = \sum_{i=1}^{k} F_k(P)$ and define

$$G_{k,n} = \max\{G_k(P)\colon P \text{ is an } n\text{-sequence}\}.$$

Our result clearly follows from the following.

Claim 2. For $k<n/2$, $G_{k,n} \leqslant n \cdot k$.

Note that since $n \cdot k \leqslant g_{k,n} \leqslant G_{k,n}$ for $k<n/2$, the last claim implies;

THEOREM 3. *For* $k<n/2$, $g_{k,n}=G_{k,n}=k \cdot n$.

As shown above every simple configuration is associated with an n-sequence. The converse, however, is not true (see [GP1]). Hence Theorem 3 is more general than Theorem 1.

Proof of Claim 2. Let b be a fixed point. The total number of switches involving b is precisely $2n-2$ (twice with any other point). If b occurs in a switch in position $i \in (1, 2, \ldots k)$ it also occurs in a switch in position $n-i$. If $i<j<n-i$ then, by continuity, b occurs in at least two switches in position j (one somewhere between the switch in position i and this in position $n-i$ and one somewhere between the switch in position $n-i$ and this in position i). Thus, any point occurs in at most $2n-2-2(n-2k-1)=4k$ switches in positions $\{1, 2, \ldots, k\} \cup \{n-k, \ldots, n-1\}$. The total number of switches in these positions is half of the sum of occurrences of points in such switches, i.e., $\leqslant \frac{1}{2} \cdot n \cdot 4k = 2nk$. The total number of switches in the first k positions is precisely half of this quantity, i.e., $\leqslant n \cdot k$. This completes the proof of Claim 2 and hence of Theorems 1 and 3. ∎

Remarks. 1. Let S be a set of n points in general position in the plane and suppose $k<n/2$. For $a, b \in S$ let $l=l(\mathrm{a}, \mathrm{b})$ be the directed line from a to b and let $N^+(l)$ denote the number of points of S in its positive side. Erdös, Lovász, Simmons and Straus [ELSS] denoted by G_k the directed graph on the set of vertices S whose edges are all segments ab, where $a, b \in S$ and $N^+(l(a, b))=k$. One can easily check that the number of k-sets of S is precisely the number of edges of G_{k-1} (=number of edges of G_{n-k-1}). It is also easy to see (analogously to the proof of Lemma 3.1 of [ELSS]) that if $a \in S$ is incident with an edge of G_i and $i<j<n-2-i$ then a is also incident with at least two edges of G_j. Thus the total number of edges incident with a in $G_0 \cup G_1 \cup \cdots \cup G_{k-1} \cup G_{n-k-1} \cup \cdots \cup G_{n-2}$ is at most $2n-2-2(n-2k-1)=4k$. Therefore the total number of edges of $G_1 \cup \cdots \cup G_{k-1}$ is $\leqslant n \cdot k$. This yields another proof of Theorem 1 (but not of the more general Theorem 3).

2. The problem of determining or estimating $f_{k,n}=\max\{f_k(S) : S$ is a configuration of n points in the plane$\}$ is much more difficult than the corresponding one for $g_{k,n}$. However, as is easily checked, $2 \cdot g_{n/2,n} = n(n-1)+f_{n/2,n}$ (for even n), i.e., the two problems are equivalent (and seem to be difficult) for $k=n/2$ (see [ELSS, Lo]).

By the results of Stanley [St], Lascoux and Schützenberger [LS] and Edelman and Greene [EG] there is a surprising one to one correspondence between n-sequences and Standard Young Tableaux of shape $(n-1, n-2, \ldots, 1)$ which might help in tackling this problem.

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